METHOD FOR HEAT TRANSFER CALCULATION USING ISOTHERMAL COORDINATES

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Equations for steady-state heat transfer are considered in curvilinear coordinates. The equations are shown to be simplest when one of the families of coordinates are isotherms. Conditions are obtained for which these coordinate systems and some exact solutions of the heat conduction equations must satisfy.

In designing the equipment for casting production and lining of metallurgical plants it is necessary to know the temperature distribution in the rollers, bars, and units of lining. Calculation methods and results of solution of heat transfer equations are reported in [1-4]. Sometimes when it is convenient to use curvilinear coordinates that permit separation of variables [1, 2], it is necessary to perform calculations for workpieces with curvilinear surfaces. Characteristics of eigenfunctions for some coordinate systems are given in [2]. In such problems numerical calculations are made with slowly convergent series by using ununiform difference schemes [5].

Occasionally, a solution may be obtained by the semi-inverse method by making assumptions on the form of the isotherms and then checking a possible solution. In some problems, for instance, with the axial symmetry, the form of the isotherms is known. In other cases one succeeds in obtaining the solution in closed form by using such curvilinear coordinates in which one of the families of coordinate lines are isotherms.

Here, there is some analogy with the method of solving the problem of plastic flow [6, 7], which is based on introducing coordinates in which one of the coordinate families are streamlines. Naturally, not any family of the curves described by differential functions can be isotherms; for this, the conditions described below are required.

In the curvilinear coordinates $\alpha(x, y, z)$, $\beta(x, y, z)$, $\gamma(x, y, z)$ the steady-state heat transfer equation is

$$\frac{\partial}{\partial\alpha}\left(\lambda\frac{H_2H_3}{H_1}\frac{\partial t}{\partial\alpha}\right)+\frac{\partial}{\partial\beta}\left(\lambda\frac{H_1H_3}{H_2}\frac{\partial t}{\partial\beta}\right)+\frac{\partial}{\partial\gamma}\left(\lambda\frac{H_1H_2}{H_3}\frac{\partial t}{\partial\gamma}\right)+H_1H_2H_3Q=0\,, \tag{1}$$

where t is the temperature; λ is the thermal conductivity, which can be a constant or a function of t; Q is the power of internal heat sources [2]; H_1, H_2, and H_3 are the coefficients of the first quadratic form (Lamé):

$$H_1=\sqrt{\left(\left(\frac{\partial x}{\partial\alpha}\right)^2+\left(\frac{\partial y}{\partial\alpha}\right)^2+\left(\frac{\partial z}{\partial\alpha}\right)^2\right)} \tag{2}$$

(the coefficients H_2, H_3 are determined analogously, see [2, 3]). (Nonstationary problems will be considered in a separate work.) In the two-dimensional problem the derivatives t, H_1, H_2 with respect to the variable γ are equal to zero whereas $H_3 = 1$.

Equation (1) is written most simply with such a choice of the variables α, β when one of the coordinate families, e.g., α-lines, are isotherms and at $\partial t/\partial\alpha = \partial t/\partial\gamma = 0$, $Q = 0$

$$\frac{\partial}{\partial\beta}\left[\lambda(t)\frac{H_1H_3}{H_2}\frac{\partial t}{\partial\beta}\right]=0\,. \tag{3}$$

Equation (3) has a solution only when H_1H_3/H_2 can be represented as a product of the functions α, β, i.e.,

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$$\frac{H_1 H_3}{H_2} = f(\alpha)\,\varphi(\beta)\,. \tag{4}$$

Then from (3) we have the following solution:

$$\lambda\frac{dt}{d\beta} = \frac{C_1}{\varphi(\beta)}; \quad \int\lambda dt = C_1\int\frac{d\beta}{\varphi(\beta)} + C_2\,, \tag{5}$$

where C_1, C_2 are the integration constants.

If (5) satisfies the boundary conditions, then we obtain an acceptable solution of the formulated problem. For the two-dimensional problem $H_3 = 1$ and condition (4) has the form

$$\frac{H_1}{H_2} = f(\alpha)\,\varphi(\beta)\,. \tag{6}$$

Here, the functions H_1, H_2 are related to the equation [8]

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial\beta}\right) = 0\,. \tag{7}$$

Violation of conditions (4) or (6) means that these lines cannot be isotherms at either boundary conditions. At $Q \neq 0$ in Eq. (1) the α-lines can be isotherms if condition (4) or (6) is fulfilled and $H_1H_2H_3Q$ can be represented as the product of the function $f(\alpha)$ by some function β. Below we shall consider two-dimensional and axisymmetric problems. In the latter case we may introduce the coordinates α, β in the meridian plane, and γ is the angle of rotation about the axis y. In view of the axial symmetry [7]:

$$H_1 = \sqrt{\left(\left(\frac{\partial x}{\partial\alpha}\right)^2 + \left(\frac{\partial y}{\partial\alpha}\right)^2\right)}, \quad H_2 = \sqrt{\left(\left(\frac{\partial x}{\partial\beta}\right)^2 + \left(\frac{\partial y}{\partial\beta}\right)^2\right)},$$

$$H_3 = x(\alpha, \beta)\,.$$

In polar coordinates of the two-dimensional problem

$$\alpha = \arctan\frac{y}{x}, \quad \beta = \sqrt{x^2 + y^2} \quad H_1 = \beta\,, \quad H_2 = H_3 = 1\,,$$

and condition (6) is fulfilled at $f(\alpha) = C_1$, $\varphi(\beta) = 1/C_1\beta$. If $Q = \text{const}$, then $H_1H_2Q = \beta Q$, i.e., in this case the constant β-lines can be isotherms as well [1-3]. For the three-dimensional problem with spherical symmetry

$$H_3 = x = \beta\cos\alpha\,, \quad \frac{H_1H_3}{H_2} = \beta^2\cos\alpha\,, \quad f(\alpha) = C_1\cos\alpha\,,$$

$$\varphi(\beta) = \frac{\beta^2}{C_1}, \quad \int\lambda dt = C_2 - \frac{C_1}{\beta}\,.$$

Not only circumferences but also straight lines passing through the origin of coordinates can be isotherms. In this case, in the two-dimensional problem

$$\alpha = \sqrt{x^2 + y^2}\,, \quad \beta = \arctan\frac{y}{x}, \quad H_1 = 1\,,$$

$$H_2 = \alpha\,, \quad H_3 = 1\,, \quad f(\alpha) = \frac{C_1}{\alpha}, \quad \varphi(\beta) = \frac{1}{C_1}$$

and condition (6) is fulfilled. If the dependence $\lambda(t)$ is adopted as a power series

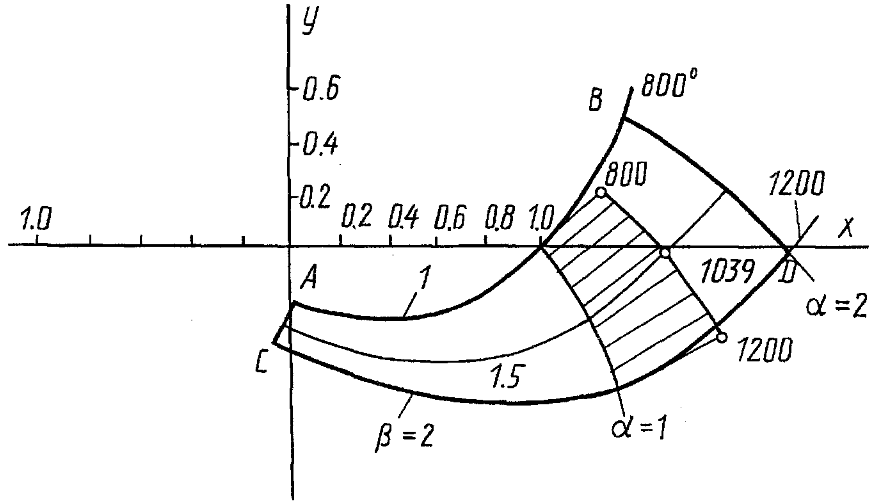

Fig. 1. Temperature distribution with respect to the thickness of the part confined by logarithmic spirals, x, y, m; t, °C.

$$\lambda(t) = \lambda_0 \left(1 + \sum_{n=1} a_n t^n\right), \tag{8}$$

where λ_0 is λ at $t = 0$ ($n = 1, 2, 3 \ldots$) and a_n are the coefficients characterizing $\lambda(t)$, then in this case

$$\lambda_0 \left(t + \sum_{n=1} \frac{a_n t^{n+1}}{n+1}\right) = C_1\beta + C_2.$$

With axial symmetry $H_1 = 1$, $H_2 = \alpha$, $H_3 = \alpha \cos\beta$, and the lines β = const describe isothermal conic surfaces. Since condition (4) is fulfilled at $f(\alpha) = C_1$, $\varphi(\beta) = \cos\beta / C_1$, we have the solution

$$\int \lambda dt = 0.5\, C_1 \ln\left(\frac{1 + \sin\beta}{1 - \sin\beta}\right) + C_2.$$

For curvilinear coordinates including two families of logarithmic spirals (in the two-dimensional problem):

$$H_1 = \sqrt{\left(\frac{\beta}{2\alpha}\right)}, \quad H_2 = \sqrt{\left(\frac{\alpha}{2\beta}\right)}, \quad \frac{H_1}{H_2} = \frac{\beta}{\alpha},$$

condition (6) is also fulfilled at $Q = 0$:

$$f(\alpha) = \frac{C_1}{\alpha}, \quad \varphi(\beta) = \frac{\beta}{C_1},$$

$$\int \lambda dt = C_1 \ln\beta + C_2.$$

(Condition (4) is not fulfilled and the logarithmic spirals fail to be the generatrices of isothermal surfaces.)

If we use formula (8), restricting ourselves only to the function $\lambda(t)$, only when $a_1 \neq 0$, then we arrive at

$$t(\beta) = \frac{1}{a_1}\left[\sqrt{\left(1 + \frac{2a_1}{\lambda_0}(C_1 \ln\beta + C_2)\right)} - 1\right].$$

Figure 1 illustrates results of the temperature distribution calculation in the region confined by the spirals

$$\beta = 1\ (AB), \quad \beta = 2\ (CD), \quad \alpha = 4.25 \cdot 10^{-2}\ (AC), \quad \alpha = 2\ (BD),$$

when temperatures $t = 800$°C on AB (at $\beta = 1$) and 1200°C on CD ($\beta = 2$) are prescribed. The calculations are made for a part of the screen manufactured from steel at $\lambda = \text{const} = \lambda_0 = 29$ W/(m·deg) and one can write $t(\beta) =$

800 + 580 ln β. The figure shows the temperature diagram along the line $\alpha = 1$. It should be noted that in some problems one can prescribe heat flux, not temperature, values, along some α-lines. Then it is more convenient to introduce, as coordinates, not isotherms, but the α-lines on which the projection of the heat flux

$$q = \frac{\lambda}{H_2} \frac{\partial t}{\partial \beta} .$$

is a constant. The logarithmic spirals represent the lines for which such a solution exists. Assuming λ = const, $t = \sqrt{\alpha}\psi(\beta)$, we obtain that q is a function only of the β-coordinate, and substitution of $t(\alpha, \beta)$ into (1) yields

$$\frac{d^2\psi}{d\beta^2} + \frac{1}{\beta}\frac{d\psi}{d\beta} + \frac{\psi}{4\beta^2} = 0 .$$

This determines the solution

$$\varphi(\beta) = C_1 \sin(0.5 \ln |\beta|) + C_2 \cos(0.5 \ln |\beta|) ,$$

$$t(\alpha, \beta) = C_1 \sqrt{\alpha} \sin(0.5 \ln |\beta|) + C_2 \sqrt{\alpha} \cos(0.5 \ln |\beta|) ,$$

$$q(\beta) = \frac{\lambda}{\sqrt{2\beta}} \left[C_1 \cos(0.5 \ln |\beta|) - C_2 \sin(0.5 \ln |\beta|) \right] .$$

We now consider the coordinates of an elliptic cylinder

$$x = \alpha\beta , \quad y = \sqrt{(\alpha^2 - 1)(1 - \beta^2)} . \tag{9}$$

Since

$$H_1 = \sqrt{\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - 1}\right)} , \quad H_2 = \sqrt{\left(\frac{\alpha^2 - \beta^2}{1 - \beta^2}\right)} , \quad H_3 = 1 ,$$

condition (6) is fulfilled at

$$f(\alpha) = C_1 (\alpha^2 - 1)^{-0.5} , \quad \varphi(\beta) = \frac{1}{C_1}\sqrt{1 - \beta^2} ,$$

therefore the hyperbolas

$$\frac{x^2}{\beta^2} - \frac{y^2}{1 - \beta^2} = 1$$

can be isotherms and in this case

$$\int \lambda dt = C_1 \arcsin \beta + C_2 .$$

Replacing β by α,

$$\alpha \text{ and } \beta , \quad x = \alpha\beta , \quad y = \sqrt{(\beta^2 - 1)(1 - \alpha^2)} \tag{10}$$

($\alpha < 1, \beta > 1$), it is easy to show that the ellipses

$$\frac{x^2}{\beta^2} + \frac{y^2}{\beta^2 - 1} = 1$$

can be isotherms

$$H_1 = \sqrt{\left(\frac{\beta^2 - \alpha^2}{1 - \alpha^2}\right)}, \quad H_2 = \sqrt{\left(\frac{\beta^2 - \alpha^2}{\beta^2 - 1}\right)}, \quad f(\alpha) = \frac{C_1}{\sqrt{1 - \alpha^2}}, \quad \varphi(\beta) = \frac{\sqrt{\beta^2 - 1}}{C_1},$$

$$\int \lambda dt = C_1 \ln (\beta + \sqrt{\beta^2 - 1}) + C_2 .$$

These formulas have been used for calculation of the temperature regimes of elliptic-section tubes manufactured from refractory materials and intended for liquid steel delivery under a meniscus level in continuous-casting moulds of thin stabs (in these cases elliptic-section tubes are used based on design considerations). Casting takes 1.0–1.5 h and already after 5–10 min the heat transfer reaches a steady state. If the temperatures t_1 and t_2 are prescribed in the β_1- and β_2-lines, then at $Q = 0$, $\lambda = \text{const}$

$$t(\beta) = t_1 + (t_2 - t_1) \ln \left(\frac{\beta + \sqrt{\beta^2 - 1}}{\beta_1 + \sqrt{\beta_1^2 - 1}}\right) \left[\ln \left(\frac{\beta_2 + \sqrt{\beta_2^2 - 1}}{\beta_1 + \sqrt{\beta_1^2 - 1}}\right)\right]^{-1} .$$

Sometimes other solutions may be found in curvilinear coordinates. Thus, if we seek a solution of Eq. (1)

$$\frac{\partial}{\partial \alpha} \left(\frac{\sqrt{1 - \alpha^2}}{\sqrt{\beta^2 - 1}} \frac{\partial t}{\partial \alpha}\right) + \frac{\partial}{\partial \beta} \left(\frac{\sqrt{\beta^2 - 1}}{\sqrt{1 - \alpha^2}} \frac{\partial t}{\partial \beta}\right) + \frac{Q(\beta^2 - \alpha^2)}{\lambda \sqrt{1 - \alpha^2} \sqrt{\beta^2 - 1}} = 0$$

(at $\lambda = \text{const}$, $Q = \text{const}$) in the form of the sum of the functions α and β, then we obtain

$$t(\alpha, \beta) = C_1 - \frac{Q}{4\lambda} \left[\alpha^2 + \arcsin^2 \alpha + \beta^2 - \ln^2 (\beta + \sqrt{\beta^2 - 1})\right] .$$

The coordinates of an elliptic cylinder are sometimes used for calculating the temperature of fuel cells of elliptic-section reactors.

For axial symmetry, condition (4) is fulfilled at coordinates (9) when $H_3 = \alpha\beta$, $\varphi(\beta) = \beta\sqrt{1-\beta^2}$:

$$t(\beta) = 0.5\, C_1 \ln \left(\frac{\sqrt{1 - \beta^2} - 1}{\sqrt{1 - \beta^2} + 1}\right) + C_2 .$$

Prescribing the temperatures on two isothermal surfaces (hyperboloid of rotation) determines the constants C_1 and C_2.

Condition (4) is also fulfilled for coordinates (10) at $\varphi(\beta) = \sqrt{b^2 - 1}$; therefore the ellipsoids of rotation can be isothermal surfaces and

$$t(\beta) = C_1 \arctan \sqrt{\beta^2 - 1} + C_2 .$$

For the axisymmetrical problem with coordinates (10) and heat release $Q = \text{const}$ in the ellipsoid volume the search for a solution in the form of a sum of the functions α and β determines

$$t(\alpha, \beta) = c_1 + C_2 \arctan \sqrt{\beta^2 - 1} + \frac{C_3}{2} \ln \left(\frac{\sqrt{1 - \alpha^2} - 1}{\sqrt{1 - \alpha^2} + 1}\right) +$$

$$+ \frac{Q}{3\lambda} \left[\ln (\alpha\beta) - 0.5 (\alpha^2 + \beta^2)\right],$$

where $H_3 = \alpha\beta$; C_1, C_2, C_3 are constants determined by the boundary conditions.

We shall consider the parabolic coordinates

$$x = \alpha\beta, \quad y = 0.5 (\beta^2 - \alpha^2), \quad H_1 = H_2 = \sqrt{\alpha^2 + \beta^2},$$

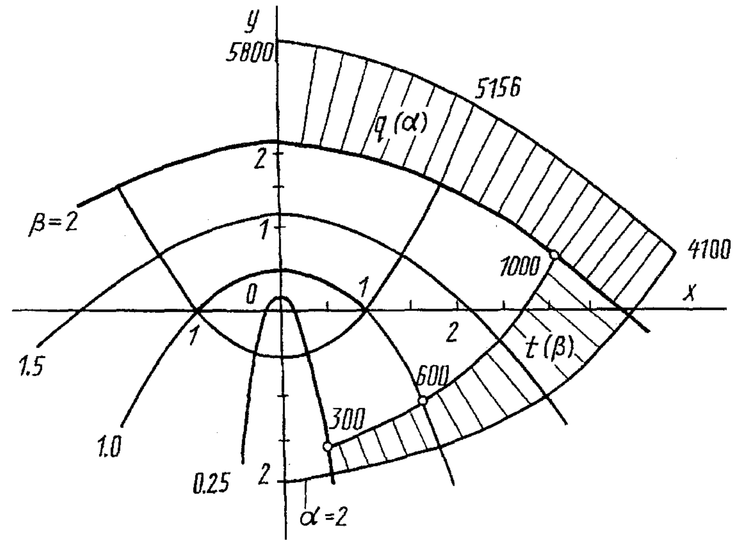

Fig. 2. Temperature distribution on the part with parabolic surfaces. The temperature on the α-line is 2°C; the heat flux q on the β-line is equal to 2 W/m^2.

when in the two-dimensional problem $H_3 = 1$, condition (6) is fulfilled and at $Q = 0$, $\varphi(\beta) = 1/C_1$ the α-lines can be isotherms

$$\int \lambda dt = C_1\beta + C_2. \tag{11}$$

Figure 2 illustrates the calculation when boundary conditions are prescribed at β_1, $t_1 = 300$°C and at β_2, $t_2 = 1000$°C, $\lambda = \text{const} = 29$ W/(m·deg). Here some isotherms ($\beta = 0.25$; 1.0; 1.5; 2.0) of the α-line = const and the temperature distribution along the α-line = 2 are shown. On the β-line = 2 a profile of the heat flux (W/m^2) is shown which is at its maximum at the top of the parabola. At $Q = \text{const} \neq 0$, $\lambda = \text{const}$ Eq. (1) has the following particular solution:

$$t(\alpha, \beta) = C_1 - \frac{Q}{12\lambda}(\alpha^4 + \beta^4).$$

It is pertinent to note that a class of coordinate systems exists (a system of parabolic corodinates also enters it) where $H_1 = H_2 = H$ in which condition (6) is always fulfilled and formula (11) yields a solution for the two-dimensional problem. From Eq. (7) it follows that ln H is the harmonic function and in addition to the solutions of (11), in which the α-lines are isotherms, a particular solution (at $Q = 0$, $\lambda = 0$, $\lambda = \text{const}$) also exists:

$$t(\alpha, \beta) = C_1 \ln H + C_2.$$

We can give examples of coordinates in which the hyperbola isotherms [6]

$$\beta = xy, \quad \alpha = 0.5(x^2 - y^2), \quad H_1 = H_2 = \frac{1}{\sqrt{2}}(\alpha^2 + \beta^2)^{-0.25}$$

and formula (11) determines the solution.

For bipolar coordinates (see [3])

$$x^2 + (y - \operatorname{ctan}\alpha)^2 = 1 + \operatorname{ctan}^2\alpha,$$

$$(x - \operatorname{cth}\beta)^2 + y^2 = \operatorname{cth}^2\beta - 1,$$

$$x = \frac{\operatorname{sh}\beta}{\operatorname{ch}\beta - \cos\alpha}, \quad y = \frac{\sin\alpha}{\operatorname{ch}\beta - \cos\alpha} \quad H = \frac{1}{\operatorname{ch}\beta - \cos\alpha}$$

and the solution is also determined by formula (11).

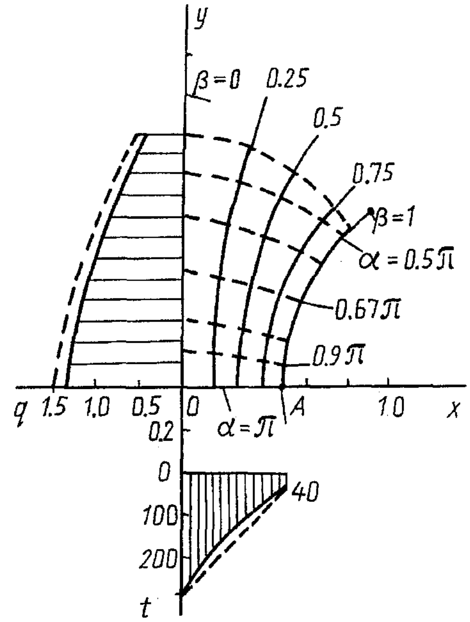

Fig. 3. Calculation using bipolar coordinates. The temperature diagram on AO is given in $^{\circ}$C; the heat flux at $\beta = 0$ is given in W/m^2.

Figure 3 illustrates the calculation of the two-dimensional problem when temperatures $t = 40^{\circ}$C at $\beta = 1$ and 300°C at $\beta = 0$ are prescribed. Moreover, the quantities $Q = 0$ are prescribed

$$\lambda (t) = \lambda_0 (1 - 5\cdot 10^{-4} t), \quad \lambda_0 = 300 \quad \mathrm{W/(m\cdot deg)}, \quad a_1 = 5\cdot 10^{-4} \quad \mathrm{deg}^{-1},$$

$$t(\beta) = 2\cdot 10^3 (1 - \sqrt{0.7225 + 0.238\beta})$$

in accordance with formula (11).

The figure shows some isotherms ($\beta = 0.25$; 0.5; 0.75) and the α-lines = const (0.47π; 0.5π; 0.58π; 0.67π; 0.77π; 0.9π), which are dashed. Furthermore, temperature variation in the section OA ($\alpha = \pi$) and a profile of the heat flux on the line $\beta = 0$ are shown. On both diagrams the dashed line indicates the lines calculated at $a_1 = 0$, λ = const. In this case, it is seen that the influence of $\lambda(t)$ variation does not exceed 12%.

In the coordinate systems in which $H_1 = H_2 = H$, condition (6), but not (4), is always fulfilled, therefore in three-dimensional problems with axial symmetry when $H_3 = x$, the α-lines can fail to be isotherms. It is easy to show that condition (4) is not fulfilled in hyperbolic and bipolar systems therefore the coordinate surfaces (the lines in the meridian plane) in them are not isotherms. At the same time for the parabolic coordinate system

$$\frac{H_1 H_3}{H_2} = \alpha\beta, \quad f(\alpha) = C_1 \alpha, \quad \varphi(\beta) = \frac{1}{C_1}\beta,$$

and a solution, in which the paraboloids are isotherms, exists and is determined by formula (9). In this system one can construct particular solutions in which the α-lines are not isotherms, for instance:

$$t(\alpha, \beta) = C_1 - \frac{Q}{4\lambda}\alpha^2\beta^2; \quad t(\alpha, \beta) = C_1 - \frac{Q}{16\lambda}(\alpha^4 + \beta^4).$$

In bipolar, hyperbolic coordinates condition (6) is fulfilled but not (4). We may also provide opposite examples. If one introduces the coordinates $\alpha = x^2 y$, $\beta = y^2 - 0.5x^2$, one has $H_1 = (x\sqrt{x^2 + 4y^2})^{-1}$, $H_2 = (\sqrt{x^2 + 4y^2})^{-1}$, $H_1 H_3 / H_2 = 1$, and condition (4) is fulfilled at $\varphi(\beta)$ = const.

Formula (11) determines the solution, and the lines (in the meriduan plane) β = const, $y = \sqrt{0.5x^2 + \beta}$ are the generatrices of the isothermal surfaces. At the same time, in the two-dimensional problem at $H_3 = 1$, $H_1/H_2 = 1/x$ and condition (6) is not fulfilled.

An answer to the question whether the given family of isotherms are isotherms depends on the fulfilment of conditions (4) or (6), i.e., on whether the combination of the Lamé coefficients may be represented as the product of the functions α and β or not. In some problems the method discussed allows solutions to be obtained in a closed form. Some problems on diffusion may be treated analogously.

NOTATION

a_1, a_2, ..., a_n, coefficients determining the temperature dependence of thermal conductivity (see formula (8)); $f(\alpha)$, function of the α-coordinate (see formula (4)); H_1, H_2, H_3, coefficients of the first differential form (Lamé coefficients) (see formula (2)); n, number of a term of the series in formula (8); q, heat flux; Q, power of volume heat release; x, y, z, Cartesian coordinates; α, β, γ, general curvilinear orthogonal coordinates; β_1, β_2, coordinates of the boundary surfaces on which the temperatures are prescribed; $\varkappa_{01}$, thermal conductivity at $t = 0$; $\varphi(\beta)$, function of the β-coordinate (see formula (4)); $\psi(\beta)$, β-function determining temperature distribution in the case of constant heat flux along the coordinate lines.

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